Modeling and Simulation

CSE Written Qualifying Exam

Spring 2026

Instructions

- This is a **CLOSED BOOK** exam. No books or notes are allowed.
- Please answer three of the following four questions. All questions are graded on a scale of 10. If you answer all four, all answers will be graded and the three lowest scores will be used in computing your total.
- Please write clearly and concisely, explain your reasoning, and show all work. Points will be awarded for clarity as well as correctness.

1 Problem 1

The invention of the laser (Light Amplification by Stimulated Emission of Radiation) represents a foundational achievement in modern physics. The principle of stimulated emission was first introduced by Albert Einstein in 1917. For their theoretical and experimental contributions to the maser and laser, Charles Townes, Nikolay Basov, and Alexander Prokhorov were awarded the Nobel Prize in Physics in 1964. Their work established the population-based rate equation framework that is still used today to model laser dynamics.

One of the simplest theoretical descriptions of a laser is the two-level system, in which atoms transition between a ground state and an excited state. While overly idealized, this model is useful for understanding the fundamental limitations of laser operation. Consider a two-level laser medium with: A ground state with population $N_1(t)$ and an excited state with population $N_2(t)$. Atoms are excited from the ground state to the excited state by an external pump at rate R (in units of atoms per second). Atoms in the excited state can decay back to the ground state through: Spontaneous emission at rate A_{21} (in s^{-1}), and stimulated emission at a rate proportional to the photon density $n_\gamma(t)$ in the cavity. Stimulated emission is modeled with a coefficient σ (with units cm^2), which reflects the cross-section of the transition. The cavity lifetime is represented by τ_c (in seconds), the average time a photon remains in the cavity before being lost. Assume that the total number of atoms N is constant: $N = N_1(t) + N_2(t)$

(a) Derive the differential equation governing the time evolution of the excited-state population $N_2(t)$, accounting for pumping, spontaneous emission, and stimulated emission.

(b) Under what conditions can population inversion ($N_2 > N_1$) occur at steady state in this two-level system?
Although such population inversion is possible mathematically, it requires a sufficiently large pumping rate. In practice, models used today introduce a third level in the laser system to achieve inversion.

(c) Write a coupled system of differential equations for $N_2(t)$ and $n_\gamma(t)$, where $n_\gamma(t)$ is the photon density in the cavity. Identify the condition under which the photon population grows exponentially, indicating the onset of lasing. This gives rise to the lasing threshold.

2 Problem 2

Consider the one-dimensional dynamical system given by the following differential equation, where $r \in \mathbb{R}$ is a control parameter:

$$\dot{x} = rx + x^3 - x^5$$

A. Fixed Points and Stability

1. Find all possible fixed points x^* of the system in terms of the parameter r.

2. Determine the linear stability of the fixed point at the origin ($x^* = 0$) as a function of r.

3. Calculate the critical value r_c at which the stability of the origin changes.

B. Bifurcation Analysis

1. For what range of r do non-zero fixed points exist? Solve for these fixed points explicitly using the substitution $u = x^2$.

2. Identify the specific type of bifurcation occurring at $r = 0$. Explain why the presence of the $+x^3$ term (as opposed to $-x^3$) classifies this bifurcation as subcritical.

3. This system exhibits additional bifurcations besides the one at $r = 0$. Find the value of r at which these occur and identify their type (e.g., saddle-node, transcritical, etc.).

C. Potentials and Global Behavior

1. The system can be expressed as a gradient flow $\dot{x} = -\frac{dV}{dx}$. Derive the potential function $V(x)$.

2. Sketch the bifurcation diagram for this system, plotting x^* vs. r. Use solid lines for stable branches and dashed lines for unstable branches.

3 Problem 3

Consider the initial value problem with periodic boundary conditions

$$u_t + f(u)_x = \varepsilon u_{xx}, \qquad x \in [0,1], \qquad u(x,0) = u_0(x),$$

where $f \in C^1(\mathbb{R})$, $f'(u) > 0$, and $\varepsilon \geq 0$. Let $x_j = j\Delta x$, $t^n = n\Delta t$, and denote the numerical approximation by u_j^n. Define $\lambda = \Delta t/\Delta x$.

(a) Construct an **explicit conservative scheme** using (i) an **upwind** discretization for the convective term $f(u)_x$ and (ii) a **centered finite difference** discretization for the diffusive term u_{xx}. Write the resulting scheme in flux form and explicitly specify the numerical flux, which should include the contribution from the diffusion term.

(b) Assume $f(u) = 0$, so that the equation reduces to

$$u_t = \varepsilon u_{xx}.$$

Using the scheme from part (1), perform a von Neumann stability analysis and derive a condition on Δt, Δx, and ε that guarantees stability.

(c) For hyperbolic conservation laws, monotonicity-preserving schemes play an important role, as they prevent the formation of spurious oscillations and ensure convergence to the physical entropy solution.

Now assume $\varepsilon = 0$ and consider the scalar conservation law

$$u_t + f(u)_x = 0.$$

You are given the following definitions.

- **Lax–Friedrichs scheme:**

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right).$$

where $\lambda = \Delta t/\Delta x$. The scheme can be written as an update map

$$u_j^{n+1} = H_j\left(u_{j-1}^n,\, u_{j+1}^n\right), \qquad H_j(a,b) = \frac{1}{2}(a+b) - \frac{\lambda}{2}\big(f(b) - f(a)\big),$$

or, in vector form, $u^{n+1} = H(u^n)$.

- **Monotone scheme:** A scheme written as $u^{n+1} = H(u^n)$ is called monotone if each component H_j is a nondecreasing function of each of its arguments.
- **Monotonicity-preserving scheme:** The scheme is monotonicity preserving if for any grid vectors v, w,

$$v \leq w \text{ componentwise} \quad \Longrightarrow \quad H(v) \leq H(w) \text{ componentwise.}$$

(c.1) Show that the Lax–Friedrichs scheme is **monotone** under a suitable CFL condition involving λ and $\max|f'(u)|$.

(c.2) Under the same condition, prove that the Lax–Friedrichs scheme is **monotonicity-preserving**.

4 Problem 4

You are designing a simulation of a national air traffic network. The model includes multiple airports, each functioning as a resource-constrained server. You must address the system architecture, output analysis, and parallel scaling.

(a) World Views
Contrast how an entity (e.g., an aircraft) "waits" for a resource in a **Process-Oriented** view versus an **Event-Oriented** view. Which of these views requires a threading or co-routine mechanism, and why?

(b) Output Analysis
When analyzing the average waiting time in a steady-state simulation, why is it necessary to identify and remove a "Warm-up Period"? In Welch's Moving Average Method, what is the specific trade-off when selecting the smoothing window size (w)?

(c) Parallel Synchronization
You partition the simulation across multiple logical processes (LPs) using the conservative **Chandy/Misra/Bryant (CMB)** algorithm. Explain why a cycle of LPs (e.g., Airport A waits for B, B for A) can result in **deadlock** if only standard messages are used. How do **Null Messages** and **Lookahead** specifically resolve this?