CSE Qualifying Exam, Fall 2024:

Numerical Methods

Instructions:

- This is a CLOSED BOOK exam. No books or notes are allowed.
- No calculators, computers, phones, or internet usage allowed at any time during the exam.
- Answer three of the following four questions. All questions are graded on a scale of 10. If you answer all four, all answers will be graded and the three lowest scores will be used in computing your total.
- Show all your work and write in a readable way. Points will be awarded for correctness as well as clarity.
- Good luck!

1. (a) [4 points] Show that

$$\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is a Householder reflector for all θ. What is the geometric meaning of θ?

(b) [3 points] Householder reflectors naturally come in pairs. What is the corresponding Householder reflector to the above reflector? How do you choose between using one or the other?

(c) [3 points] Show how to use 2×2 Householder reflectors to reduce a $n \times n$ matrix A to upper bidiagonal form, using orthgonal transformations to the left and right of A.

2. Let $A \in \mathbb{R}^{mn \times mn}$ be a two by two block matrix of the form

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}. \tag{1}$$

We write $A_{i,j}^{-1} := (A_{i,j})^{-1}$.

(a) [2 pts] Compute the block-LDU factorization of A of the form

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \begin{pmatrix} I & 0 \\ L_{2,1} & I \end{pmatrix} \begin{pmatrix} D_{1,1} & 0 \\ 0 & D_{2,2} \end{pmatrix} \begin{pmatrix} I & U_{1,2} \\ 0 & I \end{pmatrix} \tag{2}$$

providing expressions for $L_{2,1}, D_{1,1}, D_{2,2}, U_{1,2}$ in terms of A.

(b) [2 pts] Use part (a) to derive an expression of the form

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \begin{pmatrix} I & \tilde{U}_{1,2} \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{D}_{1,1} & 0 \\ 0 & \tilde{D}_{2,2} \end{pmatrix} \begin{pmatrix} I & 0 \\ \tilde{L}_{2,1} & I \end{pmatrix}, \tag{3}$$

providing expressions for $\tilde{D}_{1,1}, \tilde{D}_{2,2}, \tilde{L}_{2,1}, \tilde{U}_{1,2}$ in terms of A.

(c) [2 pts] Use the above to express A^{-1} as

$$\begin{pmatrix} (A_{1,1} - A_{1,2}A_{2,2}^{-1}A_{2,1})^{-1} & -(A_{1,1} - A_{1,2}A_{2,2}^{-1}A_{2,1})^{-1}A_{1,2}A_{2,2}^{-1} \\ -A_{2,2}^{-1}A_{2,1}(A_{1,1} - A_{1,2}A_{2,2}A_{2,1})^{-1} & A_{2,2}^{-1} + A_{2,2}^{-1}A_{2,1}(A_{1,1} - A_{1,2}A_{2,2}^{-1}A_{2,1})^{-1}A_{1,2}A_{2,2}^{-1} \end{pmatrix} \tag{4}$$

(d) [2 pts] Apply the above to cleverly chosen block matrices to derive the Woodbury matrix identity

$$\left(A - UC^{-1}V\right)^{-1} = A^{-1} + A^{-1}U\left(C - VA^{-1}U\right)^{-1}VA^{-1}. \tag{5}$$

(e) [2 pts] Assume you have an efficient algorithm for solving linear systems in $M \in \mathbb{R}^m$. Use (d) to derive an efficient algorithm for solving linear systems involving the matrix $(M + uv^T)$ for $u, v \in \mathbb{R}^m$.

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \tag{6}$$

(a) [2.5] Compute the eigenvalues and eigenvectors of A.

(b) [2.5] If you apply inverse iteration to A, to which eigenvector of A will this method converge? Prove your result.

(c) [2.5] If you apply inverse iteration to A with the diagonal shift $A \to A + 2I$, to which eigenvector of A will this method converge? Prove your result.

(d) [2.5] If you apply the QR algorithm (also known as QR iteration) to A, will it converge to a diagonal or triangular matrix? Why?

4. For a matrix $A \in \mathbb{R}^{m\times m}$, the Arnoldi process starting with an arbitrary vector $b \in \mathbb{R}^m$ can be written *incompletely* as follows:

   (a) $q_1 = b/\|b\|$

   (b) for $j = 1, 2, \dots,$ do

      i. $v = Aq_j$

      ii. for $i = 1, 2, \dots, j$ do

         A. $h_{i,j} = q_i^\top v$

         B. $v = v - h_{i,j} q_i$

      iii. $q_{j+1} = v/\|v\|$

   At the end of the jth iteration, let H_j be the $j \times j$ matrix defined with the elements $h_{i,k}$ and Q_j be the $m \times j$ matrix with columns q_k, $1 \le i, k \le j$. We do not have access to A, but we can compute products of A with v in the algorithm.

   (a) [2 points] Add a line of pseudocode to the above algorithm such that $H_j \in \mathbb{R}^{j\times j}$ satisfies $H_j = Q_{j+1}^\top A Q_j$. Be very clear in where you are inserting your code.

   (b) [4 points] At the end of the for loop (ii), if v is the zero vector, we stop the algorithm. If the algorithm is stopped at iteration j, explain with justification how we approximate some eigenvalues of A using the outputs Q_j and H_j (recall that we do not have access to A itself)?

   (c) [4 points] Suppose A is invertible. Given an algorithm and its justification for approximating $A^{-1}b$ using Q_j and H_j.