CSE Qualifying Exam, Spring 2023: Numerical Methods

Instructions:

- This is a CLOSED BOOK exam. No books or notes are allowed.
- No calculators, computers, phones, or internet usage allowed at any time during the exam (except for purposes of electronic proctoring, e.g., Honorlock).
- Answer three of the following four questions. All questions are graded on a scale of 10. If you answer all four, all answers will be graded and the three lowest scores will be used in computing your total.
- Show all your work and write in a readable way. Points will be awarded for correctness as well as clarity.
- Good luck!

1. Given a nonsingular matrix A, the Jacobi iterative method for solving the linear system of equations $Ax = b$ is

$$x^{(k+1)} = x^{(k)} + D^{-1}(b - Ax^{(k)})$$

where D is the diagonal matrix consisting of the diagonal of A, and superscripts denote an iteration number. The above iterative method uses an initial approximation $x^{(0)}$. We say that the Jacobi method converges for any initial approximation if and only if the spectral radius of the iteration matrix $(I - D^{-1}A)$ is less than 1.

Prove or disprove the following statement: if A is symmetric and positive definite and the Jacobi method converges (according to the above definition), then

$$|x_i^{(k+1)} - x_i^{(k)}| \leq |x_i^{(k)} - x_i^{(k-1)}|$$

for all components x_i of the vector x and for all k.

2. Suppose $A \in \mathbb{R}^{m\times m}$ is strictly column diagonally dominant, which means that for each k,

$$|a_{kk}| > \sum_{j\neq k} |a_{jk}|.$$

   (a) Show that if Gaussian elimination with partial pivoting is applied to A, no row interchanges take place.

   (b) Give the smallest upper bound you can for $\|L\|_1$. The 1-norm is defined as $\|L\|_1 = \max_j \sum_i |L_{ij}|$.

3. Let $A \in \mathbb{R}^{m\times n}$ be a real matrix with $m \geq n$. Let its singular value decomposition be $A = U\Sigma V^T$, with orthogonal matrices $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$, and an $m \times n$ diagonal matrix Σ with entries $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

Let u_i and v_i denote the i-th column of U and V, respectively. Let A_k denote the rank-k approximation of A given by

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

(a) Prove that A_k is the best rank-k approximation to A in the spectral norm, i.e.,

$$||A - A_k||_2 = \sigma_{k+1} \leq ||A - B_k||_2$$

for any rank-k matrix $B_k \in \mathbb{R}^{m\times n}$.

(b) Prove that A_k is the best rank-k approximation to A in the Frobenius norm, i.e.,

$$||A - A_k||_F^2 = \sum_{i=k+1}^{n} \sigma_i^2 \leq ||A - B_k||_F^2$$

for any rank-k matrix $B_k \in \mathbb{R}^{m\times n}$.

4. Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a symmetric and positive-definite matrix.

(a) Provide an algorithm for computing the lower-triangular Cholesky factorization $\mathbf{L}$ of $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$. Is the $\mathbf{L}$ satisfying this equation unique? If not, what are the degrees of freedom?

(b) For $\bar{m}_1 \dots \bar{m}_k$ positive integers summing to m consider the block matrix $\bar{\mathbf{A}}$ with block sizes given by the $\bar{m}_1 \dots \bar{m}_k$. This means, that the entry $\bar{\mathbf{A}}_{ij}$ is itself an element of $\mathbb{R}^{m_i \times m_j}$, with entries taken from the corresponding indices of $\mathbf{A}$.

Show that there exists a block-lower triangular Cholesky factor $\bar{\mathbf{A}} = \bar{\mathbf{L}}\bar{\mathbf{L}}^\top$. Characterize its uniqueness properties and provide an algorithm for computing it.

(c) Let $\hat{m}_1 \dots \hat{m}_l$ be another sequence of positive integers summing to m, with $\hat{m}_l = \bar{m}_k$. Consider the corresponding block matrix $\hat{\mathbf{A}}$ and block-lower triangular Cholesky factorization $\hat{\mathbf{A}} = \hat{\mathbf{L}}\hat{\mathbf{L}}^\top$. Show that $\hat{\mathbf{L}}_{ll}\hat{\mathbf{L}}_{ll}^\top = \bar{\mathbf{L}}_{kk}\bar{\mathbf{L}}_{kk}^\top$, irrespective of the choice of $\bar{m}_1 \dots \bar{m}_k$ and $\hat{m}_1 \dots \hat{m}_l$ for all possible block-Cholesky factors (that is, despite the nonuniqueness of the block-Cholesky factor). *Hint: First try to show the result for a sequence $\hat{m}_1 \dots \hat{m}_l$ that makes the proof as simple as possible.*