CSE Qualifying Exam, Fall 2023: Numerical Analysis

Instructions:

- This is a CLOSED BOOK exam. No books or notes are allowed.
- No calculators, computers, phones, or internet usage allowed at any time during the exam (except for purposes of electronic proctoring, e.g., Honorlock).
- Answer three of the following four questions. All questions are graded on a scale of 10. If you answer all four, all answers will be graded and the three lowest scores will be used in computing your total.
- Show all your work and write in a readable way. Points will be awarded for correctness as well as clarity.
- Good luck!

1. Consider a sequence of $m \times m$ symmetric matrices, $\{A_1, A_2, \cdots, \}$. Let A be another symmetric matrix with distinct eigenvalues $\lambda_1, \cdots, \lambda_m$, with, $|\lambda_1| > |\lambda_2| > ... > |\lambda_m|$ and corresponding (orthogonal) eigenvectors $q_1, \cdots, q_m$.

   (a) Let $V^{(0)}$ be an $m \times m$ matrix with linearly independent columns $V_1^{(0)}, \cdots, V_m^{(0)}$. Define $V^{(k)} = A_k V^{(k-1)}$ for $k = 1, 2, \cdots,$. First consider the case when $A_k := A$, a constant matrix, for all k. Does the span of $V_1^{(k)}, \cdots, V_n^{(k)}$ converge with k, for each $n \leq m$? If yes, give the rate of convergence in terms of the eigenvalues of A. (2 points)

   (b) What is the numerical difficulty in carrying out the above iteration to obtain $q_1, \cdots, q_m$? Explain how this is resolved by normalizing $V^{(k)}$ above with a QR factorization at each k. Discuss how $q_1, \cdots, q_m$ are then obtained. (2 points)

   (c) Discuss how to obtain the eigenvalues of A from the above QR algorithm. How fast do the eigenvalues converge with iteration number k? (2 points)

   (d) Given a symmetric A, construct a sequence $\{A_k\}$ that is **not** constant and yields a faster rate of convergence for all eigenvalues and eigenvectors than (b) and (c). Show the rate of convergence. (2 points)

   (e) Prove the backward stability of the algorithm in (d), stating any additional assumptions needed. (2 points)

2. For $1 \leq p \leq \infty$, define the condition number of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with respect to the p-norm as

$$\mathrm{cond}_p(\mathbf{A}) := \frac{\max_{\|x\|_p=1} \|\mathbf{A}\|_p}{\min_{\|x\|_p=1} \|\mathbf{A}\|_p}. \tag{1}$$

Here, the p norm is defined as

$$\|x\|_p = \begin{cases} \left(\sum_i |x_i|^p\right)^{1/p}, & \text{for} \quad p < \infty, \\ \max_i |x_i|, & \text{else.} \end{cases} \tag{2}$$

   (a) [2.5pts] Show that for $\mathbf{A} \in \mathbb{R}^{m \times n}$,

$$\mathrm{cond}_2\left(\mathbf{A}^T\mathbf{A}\right) = \mathrm{cond}_2(\mathbf{A})^2. \tag{3}$$

   Hint: Remember to treat the case where the condition number is infinite.

   (b) [2.5pts] How would you use QR or Cholesky factorization to solve a system $\mathbf{Ax} = \mathbf{b}$ for general but nonsingular $\mathbf{A} \in \mathbb{R}^{m \times m}$. Explain both approaches and reason why one of these approaches should generally be preferred over the other.

   (c) [2.5pts] Consider the diagonal matrix $\mathbf{D} \in \mathbb{R}^{m \times m}$ of the form

$$\mathbf{D} = \begin{pmatrix} D_{11} & & \\ & \ddots & \\ & & D_{mm} \end{pmatrix}, \quad \forall 1 \leq i \leq m, D_{ii} > 0. \tag{4}$$

   Compute $\mathrm{cond}_\infty(\mathbf{D})$ and $\mathrm{cond}_1(\mathbf{D})$.

   (d) [2.5pts] By means of an example, show that the result in (a) is not true when replacing cond_2 by a general cond_p.

3. Remember that for a symmetric and positive-definite matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$, its Cholesky factor $\mathbf{L} = \mathrm{chol}(\mathbf{A})$ is the unique lower triangular matrix with positive diagonal that satisfies $\mathbf{A} = \mathbf{L}\mathbf{L}^T$. Define the Cholesky iteration as $\mathbf{A}_0 = \mathbf{A}$, $\mathbf{L}_{k+1} = \mathrm{chol}(\mathbf{A}_k)$ and $\mathbf{A}_{k+1} = \mathbf{L}_k^T \mathbf{L}_k$. In the following, assume that all eigenvalues of $\mathbf{A}$ are distinct.

   (a) [3.0] Show that for all $k \geq 0$, $\mathbf{A}_k$ has the same eigenvalues as $\mathbf{A}_{k+1}$. *Hint: Try to show that* $\mathbf{A}_k = \mathbf{B}_k^{-1} \mathbf{A}_0 \mathbf{B}_k$ *for* $\mathbf{B}_k = \mathbf{L}_1 \cdots \mathbf{L}_k$.

   (b) [3.0] Show that that for $\mathbf{b}_k$ the leading column of $\mathbf{B}_k$, $\mathbf{b}_{k+1}$ is a positive scalar multiple of $\mathbf{A}\mathbf{b}_k$.

   (c) [3.0] Use (b) to show that $\mathbf{b}_{k+1}$ converges to the eigenvector of the largest eigenvalue of $\mathbf{A}$.

   (d) [1.0] Provide a similar algorithm that is applicable to nonsymmetric problems.

4. The following two subquestions are unrelated.

   (a) [5pts] Let A be a symmetric positive definite matrix. Consider the conjugate gradient method for solving the system of equations $Ax = b$. Suppose the initial approximation x_0 is such that the initial residual $r_0 = b - Ax_0$ is parallel to an eigenvector q of A with eigenvalue μ, i.e., $r_0 = \gamma q$ where γ is a real number. Prove that the conjugate gradient method converges in one iteration.

   (b) [5pts] Let A be a nonsymmetric and nonsingular matrix with real eigenvalues. If the Arnoldi algorithm is run on A with starting vector v for k steps, prove or disprove that the resulting $k \times k$ upper Hessenberg matrix only has real eigenvalues.