CSE Qualifying Exam, Fall 2022: Numerical Analysis

Instructions:

- This is a CLOSED BOOK exam. No books or notes are allowed.
- No calculators, computers, phones, or internet usage allowed at any time during the exam (except for purposes of electronic proctoring, e.g., Honorlock).
- Answer three of the following four questions. All questions are graded on a scale of 10. If you answer all four, all answers will be graded and the three lowest scores will be used in computing your total.
- Show all your work and write in a readable way. Points will be awarded for correctness as well as clarity.
- Good luck!

1. Consider the matrix

$$A = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 5 & 7 \\ 6 & 7 & 29 \end{pmatrix}.$$

(a) Compute by hand the lower triangular Cholesky factorization satisfying $A = LL^\top$, i.e., L is lower triangular.

(b) Prove that the lower triangular Cholesky factorization $A = LL^\top$ of any (symmetric, positive definite) matrix $A \in \mathbb{R}^{m\times m}$ satisfies

$$A = \sum_{k=1}^{m} L_{:,k} \otimes L_{:,k}.$$

Here, $L_{:,k}$ denotes the k-th column of L. For $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, the tensor product $u \otimes v \in \mathbb{R}^{m\times n}$ is defined as having entries $(u \otimes v)_{ij} = u_i v_j$.

(c) Conclude from (b), or prove in any other way, that the lower triangular Cholesky factorization of a symmetric positive-definite matrix is unique.

2. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 7 & 10 \\ 2 & 1 & 3 & 5 \end{pmatrix}.$$

(a) Determine the row rank of $\mathbf{A}$. Explain how you arrived at your answer.

(b) Determine the column rank of $\mathbf{A}$.

(c) Prove that the answers to (a) and (b) must be the same.

(d) Besides computing the SVD or anything that is equivalent to computing the SVD, state a numerically stable method for determining the rank of an arbitrary matrix.

(e) Discuss the numerical difficulties associated with computing the rank of a matrix.

3. Suppose the matrix A is real and symmetric. Consider the iteration

$$X_{k+1} = X_k(3I + X_k^2)(I + 3X_k^2)^{-1}$$

starting with $X_0 = A$.

(a) For the specific example that A is a matrix with eigenvalues $\{-10, -9, \ldots -1, 0, 1, \ldots, 199, 200\}$ what are the eigenvalues of X_7? Obviously, do not compute the above expression, since calculators and computers are not allowed, but a simplified expression for the eigenvalues will do.

(b) For a real symmetric matrix A with p positive eigenvalues, show that X_k as $k \to \infty$ has p eigenvalues of value 1.

(c) If A is square but not symmetric, does the statement in (b) still hold?

4. Let A be an upper Hessenberg matrix such that its subdiagonal entries are all nonzero. Suppose that A has an eigenvalue of 0. Show that the QR algorithm converges to this eigenvalue 0 in a single iteration.