CSE Qualifying Exam, Fall 2019: Numerical Methods

Instructions:

- This is a CLOSED BOOK exam. No books or notes are allowed.
- No calculators, computers, phones, and/or internet usage allowed at any time during the exam.
- Please answer three of the following four questions. All questions are graded on a scale of 10. If you answer all four, all answers will be graded and the three lowest scores will be used in computing your total.
- Show all your work and write in a readable way. Points will be awarded for clarity as well as correctness.
- Good luck!

Questions:

1. Consider a matrix $A \in \mathbb{R}^{n\times n}$ whose columns are linearly independent.
   (a) Prove that $\|(A^T A)^{-1}A\|_2 = \frac{1}{\sigma_n}$, where σ_n is the smallest singular value of A.
   (b) Design an algorithm to find the solution of the following optimization problem,
   $$\min_{Y\in\Gamma} \|A - Y\|_2,$$
   where Γ is the set of all rank k ($k < n$) matrices in $\mathbb{R}^{n\times n}$. You must justify that your algorithm obtains the optimizer.
   (c) What is the computational cost of your algorithm?

2. Consider a linear system of equations
   $$A\vec{x} = \vec{f},$$
   where $\vec{f} \in \mathbb{R}^n$ is given, A is a 5-band symmetric $n \times n$ matrix
   $$A = \begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ c_1 & b_2 & a_3 & b_3 & c_3 & \cdots & 0 \\ 0 & c_2 & b_3 & a_4 & b_4 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & c_{n-2} & b_{n-1} & a_n \end{bmatrix}.$$
   Its entries a_i, b_i and c_i are random numbers following uniform distributions in intervals $[10, 100]$, $(0, 2)$ and $(-2, 0)$ respectively.
   (a) Prove that A is invertible.
   (b) Write the algorithm that uses Gauss-Seidel iteration to solve the linear system.
   (c) Is Gauss-Seidel iteration convergent? If your answer is yes, prove it. If not, explain why.

3. Consider the matrix A which can be written as the product of three matrices,

$$A = \begin{bmatrix} -0.6612 & -0.4121 & -0.6269 \\ -0.6742 & -0.0400 & 0.7375 \\ -0.3290 & 0.9103 & -0.2513 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -0.6557 & -0.3056 & 0.6904 \\ -0.5848 & -0.3730 & -0.7204 \\ -0.4777 & 0.8761 & -0.0658 \end{bmatrix}$$

where the first and third matrices in the product are each orthogonal matrices.

(a) What is the Frobenius norm of A?

(b) What is the null space of A?

Now consider the matrix

$$Z = \begin{bmatrix} 6 & 0 & -8 \\ 0 & 7 & 0 \\ 8 & 0 & 6 \end{bmatrix}.$$

(c) Write the rank 2 matrix B that minimizes $\|Z - B\|_2$. Show your calculations.

4. Consider the *shifted* power iteration: starting with a random vector v, and for a given fixed scalar s, each iteration computes

$$v = (A - sI)v$$

(note that there is no inverse) followed by a normalization

$$v = v/\|v\|_2.$$

The Rayleigh quotient for the *original matrix* A is $\rho(v) = (v^T Av)/(v^T v)$. The Rayleigh quotient converges as the shifted power iteration progresses.

(a) Assume that A has 10 eigenvalues, $1, 2, \ldots, 10$. Consider the case of $s = 2$. Instead of a random vector, the initial vector is chosen to be the eigenvector corresponding to the eigenvalue 5 of A. In exact arithmetic, what will be the value of $\rho(v)$ after a very large number of iterations? Answer the same question if the iterations are performed on a computer with finite-precision arithmetic.

(b) Assume that A has 10 eigenvalues, $1, 2, \ldots, 10$. Starting with a random vector, if the shift is $s = 9$, what is the value of $\rho(v)$ when the algorithm has converged?

(c) Assume that A is a 2 by 2 matrix with eigenvalues 1 and 2. Starting with a random vector, for what value of s will the algorithm converge *fastest* to $\rho(v) = 1$?