CSE Qualifying Exam, Spring 2021: Numerical Analysis

Instructions:

- This is a CLOSED BOOK exam. No books or notes are allowed.
- No calculators, computers, phones, or internet usage allowed at any time during the exam (except for purposes of electronic proctoring, e.g., Honorlock).
- Answer three of the following four questions. All questions are graded on a scale of 10. If you answer all four, all answers will be graded and the three lowest scores will be used in computing your total.
- Show all your work and write in a readable way. Points will be awarded for correctness as well as clarity.
- Good luck!

Question 1: Consider the least squares problem

$$\min_x \|b - Ax\|_2^2$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1+\epsilon \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

where $\epsilon \in \mathbb{R}$.

(a) Find the normal equations and the exact least squares solution.

(b) Explain a good numerical algorithm to solve this problem. Explain why the suggested method is a good algorithm and give full details of each step of the algorithm.

Question 2:

Let A be $\mathbb{R}^{m\times n}$ and B be $\mathbb{R}^{n\times m}$.

(a) Show that the matrices

$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

are similar.

(b) Show that the nonzero eigenvalues of AB are the same as those of BA.

Question 3: Given a square matrix A and a set of m scalars c_i, $i = 1, \ldots, m$, define $A_i = A + c_i + I$ where I is the identity matrix. The number m may be large.

Suppose you wish to solve set of m linear systems

$$A_i x = b$$

that all have the same right-hand side vector, b. Give an efficient algorithm to solve this set of problems, i.e., more efficient than solving all m problems independently.

Question 4: Consider a symmetric, nonsingular matrix

$$H = \begin{bmatrix} 0 & B \\ B^T & A \end{bmatrix}$$

where B has dimensions $m \times n$ with $m \geq n$. Give a backward stable *finite* algorithm for computing the factorization

$$H = QMQ^T$$

where Q is orthogonal and M has the form

$$M = \begin{bmatrix} 0 & 0 & Y^T \\ 0 & X & Z^T \\ Y & Z & W \end{bmatrix}$$

where X is symmetric positive definite, W is symmetric, and Y is square and lower anti-triangular.

The anti-diagonal of a matrix is the diagonal from the top-right to the bottom-left. A lower anti-triangular matrix is zero above the anti-diagonal, i.e., the $r \times r$ matrix Y is lower anti-triangular if $Y_{ij} = 0$ for $i + j \leq r$.

Your algorithm must be *finite*, meaning that it completes after a fixed number of steps. In other words, your algorithm cannot compute eigenvalues or singular values.