CSE Qualifying Exam, Fall 2021: Numerical Analysis

Instructions:

- This is a CLOSED BOOK exam. No books or notes are allowed.
- No calculators, computers, phones, or internet usage allowed at any time during the exam (except for purposes of electronic proctoring, e.g., Honorlock).
- Answer three of the following four questions. All questions are graded on a scale of 10. If you answer all four, all answers will be graded and the three lowest scores will be used in computing your total.
- Show all your work and write in a readable way. Points will be awarded for correctness as well as clarity.
- Good luck!

1. Consider two real $m \times 3$ matrices A and V,

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \qquad V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

where a_1 is the first column of A, etc. You wish to construct V such that $V^T V = I$ and such that

$$\begin{aligned} \mathrm{span}(\mathrm{v}_3) &= \mathrm{span}(\mathrm{a}_3) \\ \mathrm{span}(\mathrm{v}_2, \mathrm{v}_3) &= \mathrm{span}(\mathrm{a}_2, \mathrm{a}_3) \\ \mathrm{span}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3) &= \mathrm{span}(\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3) \end{aligned}$$

(a) Write formulas for v_1, v_2, and v_3.

(b) If your formulas are executed on a computer using finite precision arithmetic, what issues may arise?

(c) What is backward stability? Are your formulas backward stable?

(d) If your formulas are backward stable, then how many floating point operations are required (count additions, subtractions, multiplications, and divisons as one operation each).

(e) If your formulas are not backward stable, can you improve your formulas, or are your formulas useless?

2. Given a nonsingular *tridiagonal* matrix A of dimensions 5×5, instead of computing an LU decomposition, one could compute the following decomposition:

$$A = MN$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ m_1 & 1 & 0 & 0 & 0 \\ 0 & m_2 & 1 & m_3 & 0 \\ 0 & 0 & 0 & 1 & m_4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad N = \begin{bmatrix} d_1 & n_1 & 0 & 0 & 0 \\ 0 & d_2 & n_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 \\ 0 & 0 & n_3 & d_4 & 0 \\ 0 & 0 & 0 & n_4 & d_5 \end{bmatrix}.$$

The matrices M and N are essentially triangular, since one can use successive substitution to solve systems of equations involving M or N.

(a) Show how to compute the unknown values indicated in M and N, i.e., how to compute the factorization.

(b) Prove that the $(3, 3)$ entry of A^{-1} is the same as $1/d_{33}$. (In fact, this result can be generalized.) Hint: first try to prove this result if you have a regular LU decomposition.

3. For $A \in \mathrm{R}^{m \times n}$, consider the SVD of A, $U^T AV = \Sigma$ and let its singular values be $\sigma_1 \geq \cdots \geq \sigma_p \geq 0$ where $p = \min\{m, n\}$.

(a) Prove $\|A\|_2 = \sigma_1$ and $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_p^2}$.

(b) Prove

$$\sigma_{\max}(A) = \max_{y \in \mathrm{R}^m, x \in \mathrm{R}^n} \frac{y^T Ax}{\|x\|_2 \|y\|_2}.$$

4. Suppose $A_o \in \mathrm{R}^{n \times n}$ is symmetric and positive definite and consider the following iteration:

$$
\begin{array}{ll}
\textbf{for} & k = 1, 2, \ldots \\
 & A_{k-1} = G_k G_k^T \text{ (Cholesky factorization)} \\
 & A_k = G_k^T G_k \\
\textbf{end} &
\end{array}
$$

(a) Show that this iteration is defined.

(b) Show that if $A_o = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $a \geq c$ has eigenvalues $\lambda_1 \geq \lambda_2 > 0$, then the A_k converge to $diag(\lambda_1, \lambda_2)$.